LSU Dept. of Physics and Astronomy
Qualifying Exam
Quantum Mechanics Question Bank
(01/2017)

1. Indicate whether the following statements are true or false. Do not give your response just by guessing because a correct answer will draw 1 point, an incorrect one -1 point and no answer will count for zero points.

   i) The WKB method was devised for rapidly varying potentials.

   ii) The energy of a one-dimensional harmonic oscillator perturbed by a potential linear in x can be calculated exactly.

   iii) The spin-orbit interaction in atoms is relativistic in origin.

   iv) If the spin-orbit interaction is neglected, the z component of the angular momenta of individual electrons in an atom would each be a good quantum number but the z component of the total angular momentum of all the electrons would not.

   v) The separation between the ${}^2P_{1/2}$ and ${}^2P_{3/2}$ levels in the hydrogen atom is due to radiative effects and is called the Lamb shift.

   vi) s-wave scattering is most significant for large kinetic energies of the incident particles.

   vii) The total scattering cross-section for elastic scattering by a short-range potential is all that one needs to know to obtain all the particle wave phase shifts.

   viii) In a self-consistent one electron potential for an atom, the exchange term is attractive while the direct term is repulsive.

   ix) The Bohr orbit for a μ^- bound to a proton is very much smaller than that of an electron bound to a proton.

   x) The result that most atoms do not exhibit a first order Stark affect has to do with the law of conservation of parity.

2. Using the first Born Approximation, find the differential scattering cross section for the exponential potential $V = -V_0 e^{-r/a}$. Sketch the angular dependence of the scattering amplitude.

Hint: $\int_o^{\infty} r\sin\rho r e^{-\alpha r} dr = \dfrac{2\alpha\rho}{\left(\alpha^2+\rho^2\right)^2}$

3. A and B form a complete set of commuting linear Hermitian operators, and X and Y are linear Hermitian operators which satisfy the following commutation relations with A and B:

$$[A,X] = X + \sqrt{6}Y \qquad [B,X] = X$$
$$[A,Y] = \sqrt{6}X \qquad [B,Y] = Y \quad .$$

a) If α and β are eigenvalues of A and B, respectively, show that the only non-zero matrix elements of X and Y are

$\langle\alpha'\beta'|X|\alpha\beta\rangle$ and $\langle\alpha'\beta'|Y|\alpha\beta\rangle$ with $(\alpha' = \alpha - 2, \beta' = \beta + 1)$ or $(\alpha' = \alpha + 3, \beta' = \beta + 1)$.

b) Find the explicit relationship of the matrix elements of X to those of Y.

4. Let $|\phi_1\rangle$, $|\phi_2\rangle$, and $|\phi_3\rangle$ be defined by the equations below:

$$S^2|\phi_1\rangle = 2\hbar^2|\phi_1\rangle \qquad S_z|\phi_1\rangle = +1\hbar|\phi_1\rangle,$$

$$S^2|\phi_2\rangle = 2\hbar^2|\phi_2\rangle \qquad S_z|\phi_2\rangle = +0\hbar|\phi_2\rangle,$$

$$S^2|\phi_3\rangle = 2\hbar^2|\phi_3\rangle \qquad S_z|\phi_3\rangle = -1\hbar|\phi_3\rangle,$$

Consider an ensemble of six quantum mechanical systems; one of which is in the state $|\phi_1\rangle$, two of which are in the state $|\phi_2\rangle$, and three of which are in the state $|\phi_3\rangle$.

a) Construct explicitly the density matrix for the ensemble.

b) By direct manipulation of the density matrix and other quantum mechanical operators, calculate the ensemble average for S_z.

5. A free particle with energy "E" and spin 1/2 is traveling in the x-direction. The spin of the particle also points in the x-direction. Beginning at x = 0 there is a spin dependent potential of the form

$$V(x) = V_0(1 + \alpha\sigma_z),$$

where V_0, and α are constants, $|\alpha| < 1$, and σ_z is the usual Pauli matrix.

a) Assuming that the energy "E" is smaller than either value of V(x), calculate an expression for the reflected wave.

b) Prove that the spin of the reflected wave lies entirely in the x-y plane.

c) Calculate expectation values for σ_x and σ_y for the reflected wave. By visual inspection of your answers determine when the expectation value for σ_y will be zero. Under these conditions what will be the expectation value for σ_x? Is your answer physically plausible?

6 A charged, linear harmonic oscillator is created in its ground state at time $t=-\infty$. Immediately thereafter a weak but time dependent electric field is turned on. The field is:

$$\vec{E}(t)=\hat{i}\left(\frac{N}{\tau}\right)\frac{1}{\sqrt{\pi}}\exp\left(-t^2/\tau^2\right) ,$$

where N and τ are constants.

a) What transitions will be induced by this perturbation?

b) What is the probability that the oscillator will be found in the first excited state at $t=+\infty$?

c) Examine your answer in the limits $\tau\to 0$ and $\tau\to\infty$

$$a=\sqrt{\frac{m\omega}{2\hbar}}\left(x+\frac{ip}{m\omega}\right) \qquad a|n\rangle=\sqrt{n}|n-1\rangle$$

$$a^{\dagger}=\sqrt{\frac{m\omega}{2\hbar}}\left(x-\frac{ip}{m\omega}\right) \qquad a^{\dagger}|n\rangle=\sqrt{n+1}|n+1\rangle$$

$$\int_0^{+\infty}dx e^{-ax^2}\cos(bx)=\frac{1}{2}\left(\sqrt{\frac{\pi}{a}}\right)\exp(-b^2/4a)$$

7. a) Write down the 3×3 matrices that represent the operators L_x, L_y and L_z of angular momentum for a value of $\ell=1$ in a basis which has L_z diagonal.

b) An arbitrary rotation of the state of such a system can be described by the operator

$$U=\exp(-i\alpha L_z/\hbar)\exp(-i\beta L_y/\hbar)\exp(-i\gamma L_z/\hbar) ,$$

where (α,β,γ) are the Euler angles describing the rotation. Using a), construct a 3×3 matrix representation of U.

c) Find the expectation value, $\langle \vec{L} \rangle$, in the state which results from applying U to an initial $\ell = 1,\ m = 1$ state.

8. The quantum mechanical physical system "facetium" is known to have exactly two stationary states. The states are represented in Hilbert space by the orthonormal kets $|1\rangle$ and $|2\rangle$. First, suppose that the Hamiltonian for "facetium" were known to be:

$$H_0 = a|1\rangle\langle 1| + b|2\rangle\langle 2|$$

a) What sort of numbers are "a" and "b"? Why?

b) What are the energy eigenvalues associated with the kets $|1\rangle$ and $|2\rangle$? Now suppose that the Hamiltonian for facetium were changed by the addition of the term:

$$H' = c|1\rangle\langle 2| + d|2\rangle\langle 1|$$

to H_0. That is, suppose that the full Hamiltonian for facetium were now:

$$H = H_0 + H'.$$

c) Derive a relation between c and d.

d) Calculate the eigenkets and energy eigenvalues of the full Hamiltonian H.

9. Consider a particle confined in a two dimensional box

V(x,y) = 0   0<x<L, and   0<y<L

V(x,y) = ∞   otherwise

The eigenstates and eigenenergies are given by

$$\varphi_{np}(x,y) = \tfrac{2}{L}\sin\left(\tfrac{n\pi x}{L}\right)\sin\left(\tfrac{p\pi y}{L}\right)$$
$$E_{np} = E_1(n^2 + p^2)$$

Consider a perturbation

H′(x,y) = V_o for $\frac{L}{2} - \frac{a}{2} < y < \frac{L}{2} + \frac{a}{2}$

a) Find the first order correction to the energy of the ground state.

b) Find the first order correction to the eigenfunction of the ground state.

c) The first excited energy level has a two fold degeneracy. Find the splitting of the energy level due to the perturbation in first order.

Note: Use the approximation a<<L when evaluating integrals

10. Consider two identical spin-1/2 particles of mass m in a one-dimensional box where

$$V(x) = \begin{cases} 0 & \text{for} \quad 0 < x < L \\ \infty & x < 0 \\ \infty & x > L \end{cases}$$

The possible energy levels are

$$E = \left(n_1^2 + n_2^2\right)E_1 \text{ where } E_1 = \frac{\hbar^2\pi^2}{2mL^2} \quad .$$

Write down the properly symmetrized and normalized eigenfunctions for

a) $E = 2E_1$

b) $E = 5E_1$.

11. A particle of mass m and energy $E > 0$ is incident from the left on a potential barrier given by

$$\begin{aligned} V(x) &= 0 & x < 0 \\ V(x) &= V_o(1 - x/a) & x \geq 0 \end{aligned}$$

where V_o is an energy and a is a length.
In the limit that $E << V_o$, determine the energy dependence of the transmission probability.
Hint: You may use the semiclassical approximation in this limit. To get the energy dependence you do not have to evaluate any integrals.

12. In a certain triangular molecule, an electron is free to hop from site to site. The resulting eigentstates and eigenergies for the electron are:

$$E_1 = -2e \qquad \left|\psi_1\right\rangle = \frac{1}{\sqrt{3}}\left(\left|1\right\rangle + \left|2\right\rangle + \left|3\right\rangle\right)$$

$$E_2 = +e \qquad \left|\psi_2\right\rangle = \frac{1}{\sqrt{3}}\left(\left|1\right\rangle + e^{i2\pi/3}\left|2\right\rangle + e^{i4\pi/3}\left|3\right\rangle\right)$$

$$E_3 = +e \qquad \left|\psi_3\right\rangle = \frac{1}{\sqrt{3}}\left(\left|1\right\rangle + e^{-i2\pi/3}\left|2\right\rangle + e^{-i4\pi/3}\left|3\right\rangle\right)$$

a) At $t = 0$, the state vector is

$$\left|\psi(0)\right\rangle = \frac{1}{\sqrt{6}}\left(2\left|1\right\rangle + \left|\psi 2\right\rangle + \left|3\right\rangle\right).$$

The energy is measured. What values can be found and with what probabilities?

b) Determine $\left|\psi(t)\right\rangle$ given $\left|\psi(0)\right\rangle$ above.

13. All parts of this question call for quick, short answers.

a) Estimate the lowest energy possible when an electron is confined to a distance of 10^{-8} cm.

b) The 589 nm yellow line of sodium arises from an excited state with a lifetime for optical emission of 10^{-8} seconds. Estimate the natural width of the line.

c) Evaluate $< jm\,|J_x^2|\, jm >$.

d) Find the eigenvalues of $\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix}$.

Hint: Think of the matrix as an operator in five-dimensional space, and think in terms of subspaces.

e) What are the possible values of the total angular momentum j for a d-electron? In each these states, evaluate $\vec{\ell}\cdot\vec{s}\mid j>$.

14. A particle is moving in the x-y plane subject to a uniform magnetic field in the x-direction, expressed by a vector potential $\vec{A} = (0, Bx, 0)$.

a) What is the Hamiltonian of this system?

b) Show that the operator $\hat{p}_y$ is a constant of the motion.

c) Find the allowed energies and eigenfunctions.

d) Explain why the energies are degenerate.

15. Two identical spin –1/2 particles of mass m moving in one dimension have the Hamiltonian.

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{\lambda}{m}\delta(\vec{r}_1 - \vec{r}_2)\vec{s}_1 \cdot \vec{s}_2$$

where $(p_i,\ \vec{r}_i,\ \vec{s}_i)$ are the momentum, position and spin operators for the i-th particle.

a) What operators, besides the Hamiltonian, are constants of the motion and provide good quantum numbers for the stationary states?

b) What are the symmetry requirements for the spin and spatial wave functions?

c) If $\lambda > 0$, find the energy and quantum numbers for the bound state.

16. A particle of mass m moves in a one-dimensional potential of the form

$$V(x) = \begin{cases} Fx & x \geq 0 \\ \infty & x < 0 \end{cases}$$

a) Write down the Schrödinger equation for this problem and state the boundary conditions on the wave function.

b) Estimate the ground state energy using a variational wave function $\psi(x) = x \exp(-ax)$.

c) Estimate the ground state energy using Bohr-Sommerfeld/WKB quantization. Pay attention to the behavior near each turning point.

17. All parts of this question call for quick and brief answers

a) The H_α line of the Balmer series in hydrogen has wavelength 656.3 nm. By how much is it shifted in a similar series in deuterium? Neglect relativistic effects.

b) The muons μ^+ and μ^- bind together through the Coulomb interaction to form bound states analogous to those of the hydrogen atom. Estimate the radius of the ground state of this system. $\left(m_\mu \cong 200 m_e\right)$

c) The wave function of a particle is $\left(x^2 - y^2\right) f\left(x^2 + y^2 + z^2\right)$. If a measurement of ℓ_z were made in such a state, what values will obtain and with what probabilities?

d) What kind of multipole decay governs the transition $3d_2 \to 1s_0$ (notation: $n\ell_m$) in the hydrogen atom?

e) What is the physical basis of the selection rule that a single photon transition cannot proceed between two $j = 0$ states?

f) Can the reaction $\pi^- + \pi^+ \to \pi^0 + \pi^0$ take place if the $\pi^\pm$ come together with mutual orbital angular momentum $\ell = 1$. (Spin of the pion is zero.)

18. Two particles of masses m_1 and m_2 are restricted to move in one-dimension and have coordinates and momenta $x_i, p_i, i = 1, 2$. The Hamiltonian of the system is given by

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2} m_1 \omega^2 x_1^2 + \frac{1}{2} m_2 \omega^2 x_2^2 + \frac{1}{2} k (x_1 - x_2)^2 ,$$

where $k > 0$, and k and ω are constants.

a) In a couple of lines, state what, physically, this Hamiltonian describes.

b) Obtain the energy eigenvalues of this Hamiltonian.

c) Draw an energy level diagram for the case $k >> \dfrac{m_1 m_2}{m_1 + m_2} \omega^2$.

19. a) Write down the Hamiltonian for a charged particle (e,m) in a constant magnetic field $\vec{B}$. (Hint: minimal coupling)

b) Taking the field direction as the z-axis and adopting any convenient gauge, solve for the eigenvalues of this system.

c) What are the degeneracies of each eigenvalue E?

d) Suppose an additional electric field $\vec{F}$ is applied, parallel to $\vec{B}$. What is the new Hamiltonian and how are the eigenvalues modified?

20. A quantum-mechanical physical system of spin $\hbar/2$ is described by a wavefunction $\psi(\vec{r})$ for which

$$J_z\psi(\vec{r}) = +\tfrac{\hbar}{2}\psi(\vec{r})$$

$$L^2\psi(\vec{r}) = +2\hbar^2\psi(\vec{r})$$

$|\psi(\vec{r})|^2$ is independent of θ and ϕ.

$$J^2\psi(\vec{r}) = (\text{constant})\,\psi(\vec{r})$$

a) Derive an expression for $\psi(\vec{r})$, denoting the normalized radial wave function by $R(r)$.

b) Evaluate $\langle\psi|J^2|\psi\rangle$.

21. Use the WKB quantization condition to estimate the energy eigenvalues of the potential

$$\mathrm{V}(x) = \infty\,,\ x \le 0$$
$$\mathrm{V}(x) = kx^4\,,\ x > 0$$

Explain, in a sentence or two, what is the physical assumption inherent in the WKB approximation.

Note: $\int_0^1 \sqrt{1-x^4}\,dx \approx 0.874$

22. a) Let $\hat{C}$ denote the operator that changes a function into its complex conjugate.

$$\hat{C}\psi = \psi^*$$

i. Calculate the eigenvalues of $\hat{C}$.

ii. Is $\hat{C}$ Hermitian? Prove your answer.

b) Prove that the expectation value of the square of a Hermitian operator is real and non-negative.

c) If $\hat{\Omega}$ and $\hat{\lambda}$ are both Hermitian operators, what can be said about the expectation value of their commutator? Prove your answer.

23. Using $\Psi(x) = Ne^{-\lambda x^2}$ as a trial wavefunction, use the variational principle to estimate the energy of the ground state of a one-dimensional harmonic oscillator. Write down the properly normalized wave function for the ground state.

Note: $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$ $\qquad$ $\int_{-\infty}^{+\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

24. a) Write down the Pauli spin matrices σ_i.

b) Starting with the commutation relations obeyed by all angular momentum operators, prove that

$$\sigma_i \sigma_j = i\, \varepsilon_{ijk} \sigma_k + \delta_{ij} I \,.$$

c) Let $\vec{A}$ and $\vec{B}$ be operators that commute with the Pauli spin matrices. Prove that

$$\left(\vec{\sigma} \bullet \vec{A}\right)\left(\vec{\sigma} \bullet \vec{B}\right) = \vec{A} \bullet \vec{B} + i\vec{\sigma} \bullet \left(\vec{A} \times \vec{B}\right).$$

25. None of the questions below involves extensive computation, and most may be answered in a few words. In particular questions 1-4 are either true or false. If you should respond to any of these four questions with "true", then no explanation is necessary. If you should respond with "false", then support your answer with a mathematical proof or a counterexample.

(1) All hermitian operators representing physical observables have inverses.

(2) Unitary operators constitute a subset of Hermitian operators; that is all unitary operators are Hermitian, but not all hermitian operators are unitary.

(3) The product of two unitary operators is unitary.

(4) Let $\hat{\Omega}$ and $\hat{\Lambda}$ be commuting hermitian operators. Then all eigenfunctions of $\hat{\Omega}$ are simultaneously eigenfunctions of $\hat{\Lambda}$.

(5) Do the operations of inversion and conjugation commute? That is, assuming that the operator $\hat{L}$ possesses an inverse, does $\left(\hat{\Lambda}^{-1}\right)^{\dagger} = \left(\hat{\Lambda}^{+}\right)^{-1}$?

26. The ground state wave function of the hydrogen atom, with potential energy

$$V = -\frac{e^2}{r}$$

has the functional form

$$\psi = Ne^{-r/a_0},$$

with the Bohr radius a_0, and N a normalization factor.

a) Using the s wave radial Schrödinger equation, find a_0 and the energy E in terms of the mass m, charge e, and $\hbar$.

b) Normalize the wave function to represent one particle, determining N. Calculate the electric current density.

c) Given that the proton radius is a factor 10^{-5} smaller than the Bohr radius, calculate the probability of finding the electron in the nucleus. Give your result to one significant figure.

27. A spin - $\frac{1}{2}$ charged particle is in the $s_z = -\frac{\hbar}{2}$ state at t = 0 in a constant magnetic field $\vec{B}_0 = (0,0,B_0)$. A weak, rotating magnetic field $\vec{B}_1 = (B_1 \cos \omega t, B_1 \sin \omega t, 0)$ is switched on, with $B_1 << B_o$.

a) Calculate the probability that the spin state is $s_z = \frac{\hbar}{2}$ at time t.

b) When would you expect perturbation theory to break down?

HINT: It will be useful to work with the operators $s_\pm$ when evaluating matrix elements.

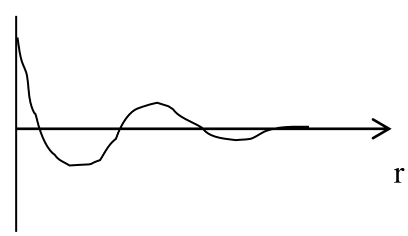
28. a) Evaluate the elastic differential scattering cross-section in the first Born approximation for the scattering of particles of mass m from a Yukawa potential $\frac{A}{r}e^{-\alpha r}$. Express your results in terms of the momentum transfer $\vec{q}$.

b) Also compute the total cross-section.

c) What happens to these results as $\alpha \to 0$?

29. All parts of this question call for quick and short answers.

a) Identify the state of the hydrogen atom whose radial function is



b) Identify which of the operators $\{L^2, L_Z, parity\}$ have sharp eigenvalues in the state described by $\frac{1}{\sqrt{6}}Y_1^1 + \frac{1}{\sqrt{6}}Y_3^1 + \sqrt{\frac{2}{3}}Y_5^1$, where Y_ℓ^m is a standard spherical harmonic.

c) Estimate the radius of the n=100 state of the hydrogen atom.

d) If the natural width of an atomic state is 1μeV, estimate its lifetime.

e) Which of the following operators is Hermitian:

$$i\frac{\partial^3}{\partial x^3},\; x\left(p^2+x^2\right)x,\; \left(x\frac{\partial}{\partial x}-\frac{\partial}{\partial x}x\right),\; \vec{x}\times\vec{p}$$

f) Which of the following are possible variational trial functions for the ground state of a one-dimensional potential well:

$$e^{-\alpha x},\; \frac{1}{x^2+\alpha^2},\; e^{-\alpha x^2}.$$

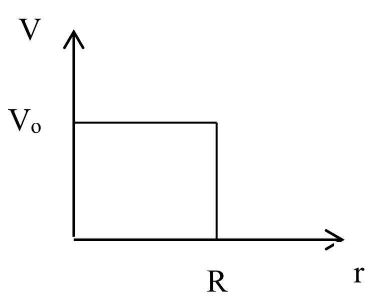
g) A one-dimensional potential well is perturbed as indicated by the dashed lines. In each case, is the ground state lowered or raised and at what order of perturbation?

h) An excited nuclear state of ${}^{8}_{4}Be$ has spin 1 and sufficient energy to decay into two α-particles (spin of α=0). Is such a decay allowed?

30. a) A spin-1/2 electron is in a uniform magnetic field $\vec{B}_o=B_o\hat{z}$. At time t = 0 the spin is pointing in the x-direction, i.e., $\left|S_x(t{=}0)\right\rangle=\frac{\hbar}{2}$. The gyromagnetic ratio is γ, and a reference frequency is defined by $\omega_o\equiv\gamma B_o$. Calculate the expectation value $\left\langle S_x(t)\right\rangle$ at time t.

b) An additional magnetic field $\vec{B}_1=B_1\left[\cos(\omega_o t)\hat{x}+\sin(\omega_o t)\hat{y}\right]$ is now applied. If an electron in the combined field $\vec{B}_o+\vec{B}_1$ has spin pointing along $+\hat{z}$ at time t=0, what is the probability that it will have flipped to $-\hat{z}$ at time t?

31. Consider scattering from a spherical hard core potential,



(V0 -> ∞)

a) Derive an expression for the phase shift, $\delta_\ell(k)$.

b) Find the leading behavior of $\delta_\ell(k)$ as a function of k for small k.

c) Consider $\ell=0$. Find the scattering length, a, and, assuming $\ell=0$ dominates, the total cross-section, σ_{tot}.

32. Consider the elastic scattering of a particle with energy $E=\frac{\hbar^2 k^2}{2m}$ from the soft sphere potential:

$$V(r)=V_o \qquad r\leq a$$

$$V(r)=0 \qquad r>a \ .$$

We want to calculate the scattering amplitude using the first Born approximation:

$$f_k^B(\theta)=-\left(\frac{2m}{\hbar^2}\right)\frac{1}{4\pi}\int d^3r' e^{i\vec{q}\cdot\vec{r}'}V(\vec{r}') \ ,$$

where the momentum transfer $\vec{q}=\vec{k}'-\vec{k}, \vec{k}'$ is the final momentum, and $(k)^2=(k')^2$.

a) In terms of V_o, a and k, when can we expect the Born approximation to give accurate results? There are at least two such conditions.

b) Calculate the scattering amplitude $f_k(\theta)$ for the soft sphere potential using the first Born approximation.

c) Show that the total cross section in the low energy limit is:

$$\sigma \simeq \pi a^2\left(\frac{4}{3}\frac{ma^2V_o}{\hbar^2}\right)^2 .$$

33. A non-relativistic electron of mass m is confined to move in one dimension. Its wave function $\psi(x)$ obeys the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V(x)\psi(x)=E\psi(x) \, .$$

a) Consider a delta function barrier at $x = 0$, described by $V(x)=V_o\delta(x)$. Show that the equations which relate the wave function and its first derivative on the left (L) and right (R) sides of the potential barrier have the following form:

$$\psi_R(0)=\psi_L(0)=\psi(0)$$

$$\left[\frac{d\psi_R}{dx}\right]_{x=0}-\left[\frac{d\psi_L}{dx}\right]_{x=0}=A\psi(0)$$

b) Give an expression for A in the last equation of part (a).

c) A beam of electrons of mass m is incident on the delta function potential of part (a). The wave function on the left and right sides is written as

$$\psi_L(x)=e^{ikx}+ae^{-ikx}$$

$$\psi_R(x)=be^{ikx}$$

Which way is the beam traveling? What is the speed of the electrons?

d) Give an expression for the transmission coefficient $T = |b|^2$ as a function of A and k.

34. a) A monochromatic beam of electrons of energy 13.6 eV is defined by the wave function $\psi=(10cm^{-3/2})\exp(ikz)$, where k is the wave number. Compute the current in the beam in the form of a number of electrons per unit area per unit time.

b) The scattering amplitude in the first Born approximation for scattering from a potential $V(\vec{r})$ is given by

$$f(\theta)=-\frac{2m}{4\pi\hbar^2}\int e^{i\vec{q}\cdot\vec{r}}V(\vec{r})d\vec{r},$$

where $\hbar\vec{q}$ is the momentum transfer.

Evaluate the cross-section in this approximation for elastic scattering from a delta-function potential $B\delta(\vec{r})$. How much of this scattering is due to the s-wave?

35. A spin-1/2 particle is subject to a static magnetic field $\vec{B}$ along the x-direction. This give rise to a Hamiltonian

$$\hat{H} = -\left(\frac{eB}{m}\right)\hat{S}_x .$$

Assume the initial state is $|\chi(t=0)\rangle = |+z\rangle$.

a) Calculate $|\chi(t)\rangle$

b) Calculate $\langle \hat{S}_z \rangle$ as a function of time. What is the precession frequency?

c) Calculate the time-dependent uncertainty in $\hat{S}_z$, $(\Delta S_z)^2 = \langle \hat{S}_z^2 \rangle - \langle \hat{S}_z \rangle^2$.

36. Given

$$[J_i, J_j] = i\hbar \sum_k \varepsilon_{ijk} J_k ,$$

$$J_z|\alpha\rangle = \hbar m|\alpha\rangle,$$

$$J^2|\alpha\rangle = \hbar^2 j(j+1)|\alpha\rangle,$$

$$\langle\alpha|\alpha\rangle = 1$$

a) Calculate $\langle\alpha|J_i|\alpha\rangle$

b) Calculate $\Delta J_i \equiv \sqrt{\langle\alpha|J_i^2|\alpha\rangle - \langle\alpha|J_i|\alpha\rangle^2}$.

c) If measurements of L^2 and L_z are made on a state whose wave function is $Ae^{-br}\sin\theta\cos\varphi$, what are the possible values found?

37. A lithium atom in its ground *s* state is placed in a small static electric field ε. By spectroscopic measurement, the ground state energy is found to shift in energy according to

$$\Delta E = -\frac{1}{2}\alpha\varepsilon^2 ,$$

where α is a constant, the atomic polarizability. From the experiment it is found that $\alpha = 24$ Å^3, where Å is an Angstrom.

a) Assuming an electric dipole interaction $H' = -\hat{\mu}\varepsilon$, use the following information to derive a simple formula for α:

   - The nearest excited state to the ground state is a p state with an energy separation of $\hbar\omega$.

   -The only relevant non-vanishing dipole matrix element is $\langle p|\hat{\mu}|s\rangle = \mu_0$.

b) The spontaneous lifetime of the p state, $\tau = 27$ ns, is related to the dipole transition moment by the Einstein A coefficient,

$$A = \frac{1}{\tau} = \frac{4}{3}\frac{\mu_0^2 k^3}{\hbar},$$

   where $k = 2\pi/\lambda$ and $\lambda = 670$nm. Use this information to write a formula for α in terms of λ, τ, and the speed of light c. Evaluate the result, either with a calculator or by estimate, to show that it agrees almost exactly with the measured value of α.

38. All parts of this question call for quick and short answers.

a) Give to within a factor of 2 the radius of the n = 100 Bohr orbit in the hydrogen atom.

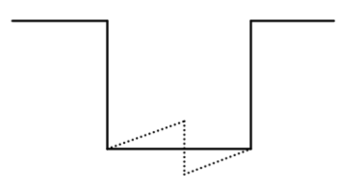
b) Give to within a factor of 2 the binding energy of the ground state when a proton and anti-proton are bound by their Coulomb interaction.

c) Identify which of the operators {L^2, L_z, parity} have definite values in the state described by $\frac{1}{\sqrt{6}}Y_{11} + \frac{1}{\sqrt{6}}Y_{31} + \sqrt{\frac{2}{3}}Y_{51}$, where the $Y_{\ell m}$ are standard spherical harmonics.

d) If the natural width of an atomic state is 1μeV, estimate its lifetime.

e) Which of the following operators is Hermitian:

$$i\frac{\partial^3}{\partial x^3},\ x(p^2+x^2)x,\ \vec{r}\times\vec{p}.$$



f) A one-dimensional potential well is perturbed as indicated by the dashed line. What order and sign of the perturbation correction do you expect for the ground state energy?

39. Consider the electromagnetic transition between the hyperfine components of the ground state of atomic hydrogen.

a) What kind of transition is it (electric or magnetic, multipolarity)?

b) The hyperfine interaction responsible can be described by $-(8\pi/3)\vec{\mu}_p\cdot\vec{\mu}_e|\psi(0)|^2$, where $\vec{\mu}_p$ and $\vec{\mu}_e$ are the magnetic moments of the particles and $|\psi(0)|^2$ the probability of finding the electron at the nucleus. Estimate the energy of the hyperfine transition.

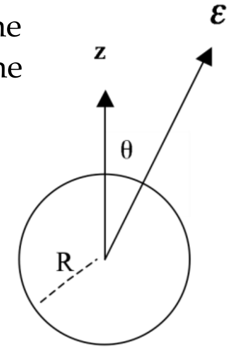
c) What is the wavelength and frequency of this radiation and comment briefly on its importance.

40. a) An electron $\left(\text{spin} = \frac{1}{2}\right)$ is prepared in an eigenstate of S_x with eigenvalue $+\hbar/2$ and subjected to a uniform magnetic field $\vec{B}=(0,0,B)$ for a time T. At that point, the field is suddenly rotated through 90° to $\vec{B}=(0,B,0)$. After another time interval T, the electron's spin in the x-direction is measured. What is the probability of obtaining the value $-\hbar/2$? Hint: It will be useful to consider S_z eigenstates for the first time interval and S_y eigenstates for the second.

b) For B = 200 G, at what earliest value of T will this probability be a maximum?

41. a) An electron is constrained to move on a circle of radius R. The dynamical variable is θ. Write down the wave functions of the eigenstates |n> of energy. What are the eigenvalues?

b) A perturbing static electric field is applied (as shown) along a diameter of the circle. Find the first non-vanishing corrections to the energy of any state |n>



Hint: What are the matrix elements of cosθ?

Caution: The first excited states need special and carefultreatment. Why?

42. All parts of this question call for quick and brief answers. You do not need to show any lengthy derivations:

a) Radio recombination lines are due to transitions $n \to n \pm 1$ in highly excited states of hydrogen. For an observed line of 6 GHz, estimate the principal quantum numbers of the states involved.

b) What kind of radiation (electric or magnetic, and order of the multipole) is observed from a transition $4f3 \to 2p1$, where these are the familiar $n\ell m$ quantum numbers of atomic states?

c) Find the eigenvalues of the matrix:

$$\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Hint: Think in terms of subspaces.

d) Can a neutral particle with spin 1 and even intrinsic parity decay into two π^0, given that its mass exceeds twice the pion mass? (Spin of $\pi^o = 0$)

e) What are the possible values of the total angular momentum of a nucleon in a d-state? If subjected to the Hamiltonian $H = A + B\vec{L} \cdot \vec{S} + C\vec{L} \cdot \vec{L}$, which of the quantities $\{S_i, L_i, J_i, S^2, L^2 \text{ and } J^2\}$ are conserved in a stationary state?

43. At time $t = 0$, a hydrogen atom is in the state

$$|t=0\rangle = \frac{1}{\sqrt{2}}|1s0\rangle - \frac{i}{3\sqrt{2}}|2p1\rangle + \frac{1}{3\sqrt{2}}|2p-1\rangle + \frac{\sqrt{7}}{3\sqrt{2}}|2p0\rangle,$$

where the kets represent normalized $|n\ell m\rangle$ states of the atom.

a) What values of angular momentum L^2 will be found upon measurement in this state?

b) What is the expectation value $\langle L^2 \rangle$ in this state?

c) If no measurements are made, what is the state $|t\rangle$ at a later time t?

d) If a measurement of L_z at $t = 0$ yields $\hbar$, what is the subsequent time evolution of the state?

e) If a weak electric field were applied at $t = 0$, will the state exhibit a linear Stark effect? Explain.

44. All parts of this question call for quick and short answers.

a) What kind of electromagnetic transition occurs between the $4f_0$ and $1s_0$ states (notation: $n\ell_m$) of the hydrogen atom?

b) The 589 nm yellow line of sodium arises from an excited state with a lifetime for optical emission of 10^{-8} s. Estimate the natural width of the line.

c) If you are told that in a certain reaction, the electron comes out with its spin always parallel to its momentum, argue that parity conservation is violated.

d) Consider three identical particles in a system which has only three states a,b and c. How many distinct allowed configurations are there if the particles are (i) bosons, (ii) fermions?

e) Electron capture by the nucleus involves the absorption by the nucleus (Z, A) of one of the atomic electrons. Which electrons (s, p, d, f, etc) would you expect to be dominantly involved? How does the capture probability scale with Z?

45. Consider a particle of mass m in a potential well

$$\begin{aligned} V(x) &= 0 \qquad && -a \le x \le a \\ &= \infty && |x| > a \end{aligned}$$

a) Using the simplest even polynomial that vanishes at $x = \pm a$, namely

$$\begin{aligned} \psi_t &= N(a^2 - x^2) \qquad && -a \le x \le a \\ &= 0 && |x| > a \end{aligned} \quad ,$$

where N is a constant, calculate variationally the ground state energy of the particle.

b) What is the exact energy for the ground state? Compare with the estimate in (a).

c) To get the first excited state in the well, what is the simplest polynomial which you would use as a trial function?

46. Each part of this question calls for brief answers.

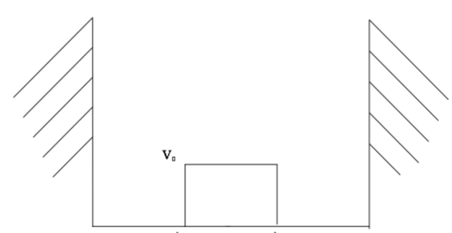
a) Estimate the shift in the wavelength (6563 Å) of the H_α line of the Balmer series in hydrogen when you go to the corresponding line in the spectrum of deuterium. Neglect relativistic effects.

b) Express the radius of the first Bohr orbit in the p-$\overline{\text{p}}$ system in terms of the Bohr radius of the hydrogen atom.

c) What order of magnitude do you expect for the elastic scattering of
i) electrons from atoms ii) neutrons from protons

d) In a partial wave analysis of the scattering of 10 MeV neutrons from deuterons, what is the highest ℓ that need be considered? Show briefly how you arrive at your estimate.

47. Give a particle of mass m in a one-dimensional square well potential with a small perturbing potential as shown in the drawing,



a) Obtain the normalized unperturbed wave functions and their energies.

b) Find the first order correction to the energy of the ground state.

c) Find the first order correction to the wave function of the ground state.

48. a) Work out the wavelength of the resonance transition (between the first excited and ground state) in the hydrogen atom. What is the common name given to this spectral line?

b) If you consider spin-orbit interactions, into how many lines does the above transition split? Without detailed or accurate calculations, what is the order of magnitude of this “fine-structure” splitting?

c) Consider next the hyperfine structure of the same transition. Into how many lines does it split?

d) How will your answers to the above parts change if you consider the deuterium atom rather than ordinary hydrogen?

49. a) The Nitrogen atom has seven electrons. Write down the electronic configuration in the ground state, and the values of parity (Π), spin (S), orbital (L) and total (J) angular momentum of the atom. (Hint: Hund’s rules).

b) If an extra electron is attached to form the N^- negative ion, what are its electron configuration and values of ${}^{2S+1}L_J^{\pi}$?

c) If now, upon photoabsorption, the extra electron is detached to leave the nitrogen atom behind in its ground state, what are the possible partial waves for the outgoing photoelectron?

50. The energy levels of the hydrogen atom from the Schrödinger equation with Coulomb potential are highly degenerate. They depend only on the radial quantum number n, but are independent of the other quantum numbers such as the orbital angular momentum L, the spin S and the total angular momentum J. However, the observed levels show small splittings between the degenerate levels. These splittings are known as fine structure, hyperfine structure and the Lamb shift.

a) Explain the physical origin of these splittings. What interactions in addition to the Coulomb interation if any are responsible for these splittings?

b) Why are these splittings small? Order these splittings in terms of the order of magnitude of their contributions.

c) Draw an energy level diagram displaying these splittings. Label each energy level with the appropriate quantum numbers, such as n, L, S, J. the ordering of the levels should be taken into account.

51. A system is in an eigenstate of the operator J^2 with eigenvalue $2\hbar$.

a) By measuring J_z additionally the pure state $|j, m_z\rangle$ is prepared. Discuss without computing explicit numbers the possible values $\hbar m_x$ of a measurement of J_x that is performed after J_z has been measured.

b) What is the probability of the values discussed in (a).

Hint: Use the $j = 1$ matrix representation for J_x given by

$$J_x = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

c) After the measurement of J_x the angular momentum component J_z is measured again. What is the probability of measuring the former values $\hbar m_z$ again?

52. The Hamiltonian of the harmonic oscillator can be written in the following form

$$H = H_0 + H_1; \; H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega x^2, \;\; H_1 = \alpha\frac{1}{2}m\omega x^2,$$

where p and x denote the corresponding momentum and position operators. The result obtained from perturbation theory can be easily compared to the exact result of the harmonic oscillator.

a) Compute the energy correction $E_n^{(1)}$ for the system in first order perturbation theory.

b) What is the value of the energy correction $E_n^{(2)}$ in second order perturbation theory?

c) Compare the energy corrections obtained in (a) and (b) with the exact energy eigenvalues of the harmonic oscillator associated with the Hamiltonian H.

53. Consider a one dimensional square well potential of width a and depth V_0. We intend to study the properties of the bound state of a particle in this well when its_width a approaches zero.

a) Show that there indeed exists only one bound state and calculate its energy E. Check that E varies with the square of the area aV_0 of the well.

b) How can the preceding considerations be applied to a particle placed in the potential $V(x) = -a\delta(x)$.

54. Consider an electron of a linear triatomic molecule formed by three equidistant atoms. We use $|\varphi_A\rangle, |\varphi_B\rangle, |\varphi_C\rangle$ to denote three orthonormal states of this electron, corresponding respectively to three wave functions localized about the nuclei of atoms *A, B, C*. We shall confine ourselves to the subspace of the state space spanned by $|\varphi_A\rangle, |\varphi_B\rangle, |\varphi_C\rangle$.

When we neglect the possibility of the electron jumping from one nucleus to another, its energy is described by the Hamiltonian H_0 whose eigenstates are the three states $|\varphi_A\rangle, |\varphi_B\rangle, |\varphi_C\rangle$ with the same eigenvalue E_0. The coupling between the states $|\varphi_A\rangle, |\varphi_B\rangle, |\varphi_C\rangle$ is described by an dditional Hamiltonian W defined by:

$$W|\varphi_A\rangle = -a|\varphi_B\rangle,$$
$$W|\varphi_B\rangle = -a|\varphi_A\rangle - a|\varphi_C\rangle,$$
$$W|\varphi_C\rangle = -a|\varphi_B\rangle,$$

where a is a real positive constant.

a) Calculate the energies and eigenstates of the Hamiltonian $H = H_0 + W$.

b) Let D be the observable whose eigenstates are $|\varphi_A\rangle, |\varphi_B\rangle, |\varphi_C\rangle$ with respective eigenvalues $-d, 0, d$. Do the operators D and H commute?

55. For a simple harmonic oscillator, consider the operator

$$\hat{O} = \frac{\hat{x}\hat{p} + \hat{p}\hat{x}}{2}$$

a) Express this operator using raising and lowering operators and show that it is Hermitian.

b) Show that the expectation value of $\hat{Q}$ in a number state is $\langle n|\hat{Q}|n\rangle = 0$.